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First-Order Intertwining Operators with Position Dependent Mass and η-Weak-Pseudo-Hermiticity Generators

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Abstract A Hermitian and an anti-Hermitian first-order intertwining operators are introduced and a class of η-weak-pseudo-Hermitian position-dependent mass (PDM) Hamiltonians are constructed. A corresponding *reference-target* η-weak-pseudo-Hermitian PDM—Hamiltonians' map is suggested. Some η-weak-pseudo-Hermitian $\mathcal{PT}$-symmetric Scarf II and periodic-type models are used as illustrative examples. *Energy-levels crossing* and *flown-away states* phenomena are reported for the resulting Scarf II spectrum. Some of the corresponding η-weak-pseudo-Hermitian Scarf II- and periodic-type-isospectral models ($\mathcal{PT}$-symmetric and non-$\mathcal{PT}$-symmetric) are given as products of the reference-target map.

1 Introduction

Recently, the growing interest in non-Hermitian Hamiltonians with real spectra [1–32] was initiated by Bender's and Boettcher's [1–6] study of the non-Hermitian Hamiltonian

$$H = p^2 + x^2(ix)^\nu; \quad \nu \geq 0. \tag{1}$$

They have observed that with properly defined boundary conditions the spectrum of (1) is real, positive and discrete. It is, thereafter, concerted that the reality of the spectrum of some non-Hermitian Hamiltonians might very well be attributed to their $\mathcal{PT}$-symmetric settings (where $\mathcal{P}$ denotes parity and $\mathcal{T}$ mimics the time-reversal). More specifically, if $\mathcal{PT}H\mathcal{PT} = H$ and if $\mathcal{PT}\Phi(x) = \pm\Phi(x)$ the eigenvalues turn out to be real. However, if the latter condition is not satisfied the eingenvalues appear in complex-conjugate pairs (cf., e.g., Ahmed in [1–6]). The Hermiticity condition as a necessity for the reality of the spectrum is relaxed therefore.

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In a series of papers [16–21], Mostafazadeh has explored the necessary and sufficient conditions for the reality of the spectra of more general diagonalizable Hamiltonians (not necessarily restricted to Hermiticity condition). He emphasized that $\mathcal{PT}$-symmetric Hamiltonians form a subclass of the so-called pseudo-Hermitian Hamiltonians [16–26]. That is, a Hamiltonian H is pseudo-Hermitian if it obeys the similarity transformation

$$\eta H \eta^{-1} = H^{\dagger},$$

where η is a Hermitian invertible linear operator and $(^{\dagger})$ denotes the adjoint.

It is now a consensus that neither Hermiticity nor $\mathcal{PT}$-symmetry serve as necessary conditions for the reality of the spectrum of a quantum Hamiltonian [16–39]. Yet, the existence of real eigenvalues is realized to be associated with a non-Hermitian Hamiltonian provided that it is an η-pseudo-Hermitian:

$$\eta H = H^{\dagger} \eta, \tag{2}$$

with respect to the nontrivial "metric" operator $\eta = O^{\dagger} O$, for some linear invertible operator $O : \mathcal{H} \to \mathcal{H}$ ($\mathcal{H}$ is the Hilbert space).

However, under some rather mild assumptions, we may even relax H to be an η-weak-pseudo-Hermitian by not restricting η to be Hermitian (cf., e.g., Bagchi and Quesne [27]), and linear and/or invertible (cf., e.g., Solombrino [28], Fityo [29], and Mustafa and Mazharimousavi [30, 31]). A classification of this sort is necessary to avoid contradictions with the well established η-pseudo-Hermiticity theorems by Mostafazadeh [16–21].

Without enforcing invertibility in the process, we have very recently introduced a class of spherically symmetric non-Hermitian Hamiltonians and their η-weak-pseudo-Hermiticity generators [32], where a generalization beyond the nodeless 1D states was proposed. We have used the same recipe and extend it for a class of η-weak-pseudo-Hermitian Hamiltonians for quantum particles endowed with position-dependent mass (PDM) [30, 31]. We were inspired by the fact that a quantum particle endowed with position-dependent mass, $M(x) = m_{\circ} m(x)$, constitutes a useful model for the study of many physical problems [33–45]. For example, they are used in the energy density many-body problem [43], in the determination of the electronic properties of semiconductors [44] and quantum dots [45], etc.

The above forms the stimuli and inspiration of this paper which is organized as follows. In Sect. 2, we introduce two first-order intertwining PDM-differential operators to construct a class of η-weak-pseudo-Hermitian PDM-Hamiltonians. A *reference* and *target* η-weak-pseudo-Hermitian PDM-Hamiltonians' map is also introduced in the same section. In Sect. 3, an η-weak-pseudo-Hermitian complexified $\mathcal{PT}$-symmetric Scarf II model is used as an illustrative *reference* model. Two η-weak-pseudo-Hermitian Scarf II-isospectral models are given as products of the *reference-target* map (for the sake of completeness of this work), in the same section. An η-weak-pseudo-Hermitian PT-symmetric reference periodic-type model is given in Sect. 4, along with one of its isospectral *reference-target* map descendants. We conclude in Sect. 5.

2 First-Order Intertwining PDM-Operator η and η-Weak-Pseudo-Hermiticity

For the sake of completeness and to keep the current study minimally self-contained, we recollect that one may avoid invertibility as a necessity for η and consider a non-Hermitian η-weak-pseudo-Hermitian Hamiltonian H satisfying the intertwining relation $\eta H = H^{\dagger} \eta$.

Then, if η_1 is Hermitian, it is easy to show that $(\eta_1 H)$ is also Hermitian (cf., e.g., [29]). On the other hand, if η_2 is anti-Hermitian such that $\eta_2 = i\eta_1$, then $(\eta_2 H)$ is anti-Hermitian too. Yet, in both cases, the Hamiltonian H may very well be classified as an η-weak-pseudo-Hermitian. The proof of the reality of the spectrum in both cases is straightforward:

Let $\psi_n(x)$ and E_n be the eigenfunctions and eigenvalues of H, respectively. Then, the Hermiticity of $(\eta_1 H)$ implies that

$$\langle\psi_n|\eta_1 H|\psi_n\rangle = E_n\langle\psi_n|\eta_1|\psi_n\rangle,$$

where

$$\langle\psi_n|\eta_1 H|\psi_n\rangle \in \mathbb{R} \ni \langle\psi_n|\eta_1|\psi_n\rangle \Longrightarrow E_n \in \mathbb{R}.$$

Next, the anti-Hermiticity of $(\eta_2 H)$ yields (with $\eta_2 = i\eta_1$)

$$\langle\psi_n|\eta_2 H|\psi_n\rangle = E_n\langle\psi_n|\eta_2|\psi_n\rangle \Rightarrow i\langle\psi_n|\eta_1 H|\psi_n\rangle = iE_n\langle\psi_n|\eta_1|\psi_n\rangle.$$

Then, $\langle\psi_n|\eta_2 H|\psi_n\rangle$ and $\langle\psi_n|\eta_2|\psi_n\rangle$ are identically pure imaginary resulting in $E_n \in \mathbb{R}$.

Hence, the reality of the spectrum for both cases is secured provided that $\langle\psi_n|\eta_i|\psi_n\rangle \neq 0$, otherwise the reality/imaginary nature of E_n remains undetermined. In the forthcoming developments we shall focus only on the Hamiltonians satisfying the above mentioned properties.

2.1 Consequences of Hermiticity and Anti-Hermiticity on a First-Order Intertwining PDM-Operator

A first-order differential operator η_1 with position-dependent mass settings:

$$\eta_1 = -i[\mu(x)\partial_x + G_1(x)] + F_1(x) \tag{3}$$

is Hermitian if and only if $G_1(x) = \mu'(x)/2$, where $\mu(x) = 1/\sqrt{M(x)}$, $M(x) = m_\circ m(x)$, and prime denotes derivative with respect to $\mathbb{R} \ni x \in (-\infty, \infty)$. On the other hand, a first-order PDM-differential operator

$$\eta_2 = \mu(x)\partial_x + G_2(x) + iF_2(x) \tag{4}$$

is anti-Hermitian if and only if $G_2(x) = \mu'(x)/2$. It is obvious that $\eta_2 = i\eta_1$. Moreover, a symmetry ordering recipe of the momentum and position dependent mass would imply a non-Hermitian Schrödinger Hamiltonian

$$H = -\partial_x\left(\frac{1}{M(x)}\right)\partial_x + V(x) + iW(x), \tag{5}$$

where $\hbar = 2m_\circ = 1$ and $\alpha = \gamma = 0$ and $\beta = -1$ in (6) of von Roos in [33–39].

Within the above settings, the intertwining relation

$$\eta_j H = H^\dagger \eta_j; \quad j = 1, 2 \tag{6}$$

(where $F_j(x), G_j(x), V(x), W(x) \in \mathbb{R}$) would imply

$$2\mu(x)^2 F_j'(x) + 2\mu(x)W(x) = 0, \tag{7}$$

$$2\mu(x)\mu'(x)F_j'(x)+\mu'(x)W(x)+\mu(x)W'(x)+\mu(x)^2F_j''(x)=0, \tag{8}$$

$$-\mu(x)V'(x)-\frac{1}{2}\mu(x)^2\mu'''(x)+2F_j(x)W(x)-\mu(x)\mu'(x)\mu''(x)=0. \tag{9}$$

Which in turn yields

$$W(x)=-\mu(x)F_j'(x), \tag{10}$$

$$V(x)=-F_j(x)^2-\frac{1}{2}\mu(x)\mu''(x)-\frac{1}{4}\mu'(x)^2+\alpha_\circ, \tag{11}$$

where $\alpha_\circ\in\mathbb{R}$ is an integration constant. Then our η-weak-pseudo-Hermitian PDM-Hamiltonian (5) reads

$$H=-\mu(x)^2\partial_x^2-2\mu(x)\mu'(x)\partial_x+\tilde{V}_j(x), \tag{12}$$

where

$$\tilde{V}_j(x)=-F_j(x)^2-\frac{1}{2}\mu(x)\mu''(x)-\frac{1}{4}\mu'(x)^2+\alpha_\circ-i\mu(x)F_j'(x), \tag{13}$$

and $\alpha_\circ\in\mathbb{R}$ may very well serve for some feasible exactly solvable η-weak-pseudo-Hermitian Hamiltonian models. Obviously, $F_j(x)$'s are our generating functions for some η-weak-pseudo-Hermitian Hamiltonians in (12).

2.2 Corresponding η-Weak-Pseudo-Hermitian Hamiltonians' *Reference-Target* Map

In this section we invest our experience in the point-canonical-transformation [30–32, 40–42] and consider our non-Hermitian η-weak-pseudo-Hermitian Hamiltonian H, in (12), in the one-dimensional Schrödinger equation:

$$(H-E)\psi(x)=0, \tag{14}$$

to construct the so-called *reference/old* and *target/new* non-Hermitian η-weak-pseudo-Hermitian Hamiltonians' map.

A substitution of the form

$$\psi(x)=\varphi(q(x))/\sqrt{\mu(x)}$$

would result in the so-called *target/new* Schrödinger equation

$$\mu(x)^2[q'(x)]^2\partial_q^2\varphi(q)+\mu(x)[\mu(x)q'(x)]'\partial_q\varphi(q)$$
$$+[F_j(x)^2+i\mu(x)F_j'(x)+E-\alpha_\circ]\varphi(q)=0. \tag{15}$$

Next, one may avoid the first-order derivative of $\varphi(q)$ to come out with the traditional form of the one-dimensional Schrödinger equation and substitute

$$q'(x)=1/\mu(x)$$

to obtain

$$-\partial_q^2\varphi(q)+[\alpha_\circ-F_j(x(q))^2-i\mu(x)F_j'(x)-E]\varphi(q)=0. \tag{16}$$

This in turn, with

$$\frac{dF_j(x)}{dx}=\frac{dF_j(q(x))}{dx}=\frac{dq(x)}{dx}\frac{dF_j(q)}{dq}=\frac{1}{\mu(x)}\frac{dF_j(q)}{dq},$$

collapses into the *reference/old* Schrödinger equation

$$-\partial_q^2\varphi(q)+[\tilde{V}_{\rm eff}(q)-E]\varphi(q)=0, \tag{17}$$

where

$$\tilde{V}_{\rm eff}(q)=\alpha_\circ-F_j(q)^2-iF_j'(q). \tag{18}$$

The *reference-target* map is therefore complete and an explicit correspondence between two bound state problems is obtained. That is, one needs to find the exact, quasi-exact, or conditionally-exact solution (eigenvalues and eigenfunctions) for the *reference/old* Schrödinger equation (17) and map it into the *target/new* Schrödinger equation (14), where H is defined in (12).

3 An η-Weak-Pseudo-Hermitian PT-Symmetric Reference Scarf II Model

A complexified PT-symmetric Scarf II potential

$$V(x)=-V_1\,\mathrm{sech}^2 q-iV_2\,\mathrm{sech}\,q\tanh q;\quad V_1>0,\ V_2\neq 0,\ V_1,V_2\in\mathbb{R}, \tag{19}$$

is studied by Bagchi and Quesne [46] using complex Lie algebras (sl(2, $\mathbb{C}$) in particular). Therein, exact real eigenvalues are reported:

$$E_{n,\epsilon}=-\left[\frac{1}{2}\left(\sqrt{V_1+\frac{1}{4}+|V_2|}+\epsilon\sqrt{V_1+\frac{1}{4}-|V_2|}\right)-n-\frac{1}{2}\right]^2,\quad \epsilon=\pm1, \tag{20}$$

where

$$n=0,1,2,\ldots<\frac{1}{2}\left(\sqrt{V_1+\frac{1}{4}+w|V_2|}+\epsilon\sqrt{V_1+\frac{1}{4}-|V_2|}-1\right).$$

On the other hand, an η-weak-pseudo-Herrmiticity generator of the form

$$F_j(q)=-V_2\,\mathrm{sech}\,q\Longrightarrow F_j'(q)=V_2\,\mathrm{sech}\,q\tanh q \tag{21}$$

would imply (with $\alpha_\circ=0$)

$$\tilde{V}_{\rm eff}(q)=-V_2^2\,\mathrm{sech}^2 q-iV_2\,\mathrm{sech}\,q\tanh q. \tag{22}$$

Then the eigenvalues in (20), with $V_1=V_2^2$, read

$$E_{n,\epsilon}=-\left[\frac{1}{2}\left(\sqrt{V_2^2+\frac{1}{4}+|V_2|}+\epsilon\sqrt{V_2^2+\frac{1}{4}-|V_2|}\right)-n-\frac{1}{2}\right]^2,\quad \epsilon=\pm1, \tag{23}$$

which collapses, with $|V_2|>1/2$ and $\epsilon=+1$, into

$$E_{n,\epsilon=+1}=-\left[|V_2|-n-\frac{1}{2}\right]^2;\quad n=0,1,2,\ldots,n_{\max}<(|V_2|-1/2). \tag{24}$$

However, for $|V_2| < 1/2$ and $\epsilon = \pm 1$, and for $|V_2| > 1/2$ and $\epsilon = -1$ one would get empty sets of energy eigenvalues.

Nevertheless, it is obvious that the phenomenon of *energy-levels crossing* is manifested (cf., e.g., Mustafa and Znojil [7–15]) in the current PT-symmetric Scarf II energy spectrum in (24). Such a phenomenon occurs when a state E_{n_1} at $|V_2| = V_{2,1}$ and a state E_{n_2} at $|V_2| = V_{2,2}$ have the same energy eigenvalues. In this case

$$E_{n_1}(|V_2| = V_{2,1}) = E_{n_2}(|V_2| = V_{2,2}); \quad n_2 > n_1,$$

and hence the energy-level crossing occurs when

$$V_{2,2} - \Delta n = V_{2,1}; \quad \Delta n = n_2 - n_1 > 0.$$

It might also be interesting to observe that the so called *flown away* states phenomenon (cf., e.g., Mustafa and Znojil [7–15]) is feasible when $|V_2| \gg n$. In this case, such states *fly away* and disappear from the spectrum.

3.1 Corresponding Scarf II-Isospectral η-Weak-Pseudo-Hermitian Models

Let us start with a general case and consider $q(x) = \pm \ln f(x);\ f(x) \in \mathbb{R}$ to imply

$$M(x) = [\partial_x \ln . f(x)]^2 \Longrightarrow f(x) = \exp\left(\pm \int^x \sqrt{M(z)}dz\right) \tag{25}$$

and

$$\tilde{V}_{\text{eff}}(x, q = \pm \ln f(x)) = -4V_2^2 \frac{f(x)^2}{(f(x)^2+1)^2} \mp 2iV_2 \frac{f(x)(f(x)^2-1)}{(f(x)^2+1)^2}. \tag{26}$$

Then, a class of Scarf II-isospectral η-weak-pseudo-Hermitian models is now generated and $f(x)$ may very well be considered as a Scarf II-isospectral PDM η-weak-pseudo-Hermiticity generator. Of course, as long as a chosen $f(x)$ generates a physically acceptable position dependent mass function $M(x)$.

Under such settings, one may conclude that the number of the related Scarf II-isospectral η-weak-pseudo-Hermitian models is large. we only choose one illustrative model.

An $f(x) = (x^2+1)$ would lead to

$$q(x) = \ln(x^2+1) = \int^x \sqrt{M(z)}dz \Longrightarrow M(x) = \frac{4x^2}{(x^2+1)^2} \tag{27}$$

and

$$\tilde{V}_{\text{eff}}(x) = -4V_2^2 \frac{(x^2+1)^2}{((x^2+1)^2+1)^2} - 2iV_2 \frac{(x^2+1)((x^2+1)^2-1)}{((x^2+1)^2+1)^2}, \tag{28}$$

which is a non-singular non-$\mathcal{PT}$-symmetric η-weak-pseudo-Hermitian model.

4 An η-Weak-Pseudo-Hermitian PT-Symmetric Reference Periodic-Type Model

An η-weak-pseudo-Herrmiticity generator of the form

$$F_j(q) = -\frac{4}{3\cos^2 q - 4} - \frac{5}{4} \tag{29}$$

would imply (with $\alpha_\circ = 0$) an effective periodic-type PT-symmetric reference potential of the form

$$\tilde{V}_{\text{eff}}(q) = \frac{1}{9}\frac{(-30\cos^2 q + 24)}{(\cos^2 q - \frac{4}{3})^2} + \frac{4i\sin 2q}{3(\cos^2 q - \frac{4}{3})^2} - \frac{25}{16}. \tag{30}$$

In a straightforward manner one may show that

$$\tilde{V}_{\text{eff}}(q) = -\frac{6}{(\cos q + 2i\sin q)^2} - \frac{25}{16}. \tag{31}$$

Such an effective potential represents a "shifted by $-\frac{25}{16}$" Samsonov's and Roy's [7–15] periodic potential model. The solution of which is reported for the interval $q \in (-\pi, \pi)$ with the boundary conditions $\varphi(-\pi) = \varphi(\pi) = 0$ as

$$\begin{aligned}\varphi(q) = \bigg\{&\big[(16 - n^2)\cos q - 2i(n^2 - 4)\sin q\big]\sin\left[\frac{n}{2}(\pi + q)\right] \\ &- 6n\sin q\cos\left[\frac{n}{2}(\pi + q)\right]\bigg\}(\cos q + 2i\sin q)^{-1}\end{aligned} \tag{32}$$

and

$$E_n = \frac{n^2}{4} - \frac{25}{16}; \quad n = 1, 3, 4, 5, \ldots \tag{33}$$

with a missing $n = 2$ state (the details of which can be found in Samsonov and Roy [7–15]).

4.1 Corresponding Periodic-Type-Isospectral η-Weak-Pseudo-Hermitian Models

One of the choices for $q(x)$ is the class of models descending from $q(x) = \arctan g(x)$; $g(x) \in \mathbb{R}$ to imply

$$M(x) = \left[\frac{\partial_x . g(x)}{1 + g(x)^2}\right]^2, \tag{34}$$

and

$$\tilde{V}_{\text{eff}}(x) = -\frac{6(g(x)^2 + 1)}{(1 + 2ig(x))^2} - \frac{25}{16}. \tag{35}$$

Then, a class of periodic-type isospectral non-$\mathcal{PT}$-symmetric (for even $g(x)$ and $\mathcal{PT}$-symmetric for odd $g(x)$) η-weak-pseudo-Hermitian models (not necessarily periodic-type) is generated, where $g(x)$ is now a periodic-type isospectral PDM η-weak-pseudo-Hermiticity generator. Again, as long as a chosen $g(x)$ generates a physically acceptable position dependent mass function $M(x)$.

To illustrate the process, let us take $g(x) = x$ then $M(x) = (1 + x^2)^{-2}$ and

$$\tilde{V}_{\text{eff}}(x) = -\frac{6(x^2 + 1)}{(1 + 2ix)^2} - \frac{25}{16}; \quad \mathbb{R} \ni x \in (-\infty, \infty). \tag{36}$$

Next, one may choose to work the other way around and start with $M(x) = (1 + x^2)^{-2}$ to come out with the same $\tilde{V}_{\text{eff}}(x)$ in (36).

5 Conclusion

In this work, we have introduced two first-order intertwining PDM-differential operators (a Hermitian η_1 and an anti-Hermitian $\eta_2 = i\eta_1$) and constructed a class of η-weak-pseudo-Hermitian PDM-Hamiltonians. We have observed that the Hermiticity of $(\eta_1 H)$ and the anti-Hermiticity of $(\eta_2 H)$ leaves the form of the resulting η-weak-pseudo-Hermitian PDM-Hamiltonian invariant (cf. (16) and/or equivalently (18)) regardless of the Hermiticity nature of our intertwining first-order differential operators η_1 and η_2. Moreover, we have used a Liouvilean-type change of variables, $q'(x) = 1/\mu(x)$, and constructed the corresponding *reference-target* η-weak-pseudo-Hermitian PDM-Hamiltonians' map. Hence, an explicit correspondence between two bound-state problems is obtained.

Within the setting of such *reference-target* map, we have generated a complexified $\mathcal{PT}$-symmetric Scarf II model and reported/fine-tuned its exact eigenvalues along with two η-weak-pseudo-Hermitian Scarf II-isospectral models. We have also generated an η-weak-pseudo-Hermitian PT-symmetric reference periodic-type model and reported one of its isospectral *reference-target* map descendants. Nevertheless, one should be reminded that the number of the isospectral *reference-target* map descendants is not only limited to one and/or two but rather remains elusive and unexplored as yet.

Under the parametric settings generated for our complexified $\mathcal{PT}$-symmetric Scarf II model, we have reported and discussed the phenomenon of *energy-levels crossing* and the feasible manifestation of *flown away states* phenomenon. The reader may refer to Mustafa and Znojil [7–15] for more details on these phenomena. Nevertheless, more comprehensive details on the energy levels-crossing phenomenon the reader may wish to refer to, e.g., Guida et al. [48] who studied the energy-levels crossing of fermionic systems in Instanton-Anti-Instanton valley, Nishino and Deguchi [49] who discussed energy-levels crossings in the one dimensional Hubbard model, and Bhattacharya and Raman [50] who elaborated on the detection of energy-levels crossings and presented an algebraic method of finding such crossings.

However, in the process of testing a complexified non-$\mathcal{PT}$-symmetric Morse model, we observed that a generating function of the form $F_j(q) = \eta e^{-q}$ would lead to

$$V_{\text{eff}}(q) = -\eta^2 e^{-2q} + i\eta e^{-q}. \tag{37}$$

The bound-state solutions of which form an empty set of eigenvalues (cf., e.g., Bagchi and Quesne [46] and Ahmed [47]). Although the potential in (26) remains an η-weak-pseudo-Hermitian, it rather represents an unfortunate *reference* model and consequently leading to a set of unfortunate isospectral η-weak-pseudo-Hermitian *target* Hamiltonians.

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